$\mathbf{99^{100} > 100^{99}}$

Proof:
Take base-10 logarithms of both sides:

$$\log(99^{100}) > \log(100^{99}) \Rightarrow 100 \log 99 > 99 \log 100$$

Since $\log 100 = 2$, this becomes:

$$100 \log 99 > 198 \Rightarrow \log 99 > 1.98$$

We estimate $\log 99$ using a secant line between known points on $\log x$: $(10, 1)$ and $(100, 2)$. The slope is

$$\frac{2-1}{100-10} = \frac{1}{90}$$

The secant line is $y = 1 + \frac{1}{90}(x - 10)$. At $x = 99$, this evaluates to

$$1 + \frac{1}{90}(99 - 10) = 1 + \frac{89}{90} = \frac{179}{90} = 2 - \frac{1}{90}$$

Which is less than 1.98.
Since $\log x$ is concave down, it lies above any secant line between two points, so:

$$\log 99 > \frac{179}{90} > 1.98 \Rightarrow 100 \log 99 > 198 \Rightarrow 99^{100} > 100^{99}$$

■